On the sharpness of an error bound for a Galerkin method to solve parabolic differential equations

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Abstract

This paper discusses the sharpness of an error bound for the standard Galerkin method for the approximate solution of a parabolic differential equation. A backward difference is used for discretization in time, and a variational method like the finite element method is considered for discretization in space. The error bound is written in terms of an averaged modulus of continuity. Whereas the direct estimate follows by standard methods, the sharpness of the bound is established by an application of a quantitative extension of the uniform boundedness principle as proposed in [W. Dickmeis, R.J. Nessel, E. van Wickeren, A general approach to counterexamples in numerical analysis, Numer. Math. 43 (1984) 249–263].

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1 Introduction

One approach to solve initial boundary value problems for parabolic differential equations numerically is to use a variational method for the elliptic part of the equation in combination with finite differences for a discretization in time (cf. [20]).

Here we discuss partial differential equations like the heat equation

$$\frac{\partial}{\partial t}u(t,\vec{x}) - c\Delta u(t,\vec{x}) = f(t,\vec{x}) \tag{1.1}$$

in a weak representation and replace the time derivative by a backward difference. The resulting Galerkin approximation is analyzed in detail in [18].

Moduli of continuity are an established means to express error bounds for initial boundary value problems that are solved by difference schemes, cf. [1] where a probabilistic interpretation of Green's functions is applied, or [7, 8] for a purely analytical approach. Also, in connection with variational methods, such moduli are used in error estimates for elliptic problems and the estimates can be established via K-functional techniques (cf. [17, 10]).

Here, we give a similar error bound based on an averaged modulus of continuity (τ-modulus) that measures the smoothness of the exact solution and determines the rate of convergence of the approximation. Averaged moduli were introduced by the Bulgarian school of Approximation Theory, see [19] and [2] for computational aspects.

Then we prove the sharpness of this estimate in terms of counterexamples. We obtain these examples by using a quantitative extension of the uniform boundedness principle. This involves a new approach to the construction of resonance elements that are a prerequisite for applying the boundedness principle.

We use the same train of thought as in our previous work that utilizes discrete Green's functions (cf. [6]). This tool was used in [3] to establish the sharpness of an error bound in connection with finite difference schemes for ordinary and later in [7, 8] for partial differential equations. It was adapted in [1]. In the present work we combine discrete Green's functions with eigenfunctions of the problem.

The treatment is restricted to those inhomogeneous problems for which solutions actually belong to appropriate Banach spaces of continuously differentiable functions.

2 Preliminaries

Let H and V, $V \subset H$, be real Hilbert spaces with inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$ such that for all $u \in V$ one has $\|u\|_H \le \|u\|_V$. Further, let $a(\cdot,\cdot)$ be a coercive (V-elliptic), bounded and symmetric bilinear form, i. e., constants $0 < c, C \in \mathbb{R}$ exist such that

$$a(u,u) \ge c\|u\|_V^2 \quad \text{for all } u \in V, \tag{2.1}$$

$$|a(u,v)| \le C\|u\|_V\|v\|_V \quad \text{for all } u,v \in V, \tag{2.2}$$

and $a(u,v) = a(v,u)$ for all $u,v \in V$.

Riesz representation theorem for Hilbert spaces states that for each bounded linear functional f^* on V and for each weak problem

$$a(u,v) = f^*(v) \quad \text{for all } v \in V. \tag{2.3}$$

there is a unique solution $u \in V$. Due to the theorem of Lax and Milgram this still is true if $a(\cdot,\cdot)$ is not symmetric as it would be the case in many applications. Indeed, symmetry of $a(\cdot,\cdot)$ will not be needed in the proof of the error estimate Theorem 3.1 but will be utilized to prove sharpness and is given in the context of (1.1).

By applying the finite element method one rewrites a differential equation into a weak problem by partial integration. Then one solves this problem in a (finite dimensional) subspace $V_h \subset V$ (Ritz-Galerkin method). So one looks for an approximate solution $u_h \in V_h$ such that

$$a(u_h,v) = f^*(v) \quad \text{for all } v \in V_h. \tag{2.4}$$

Obviously, each $u \in V$ is solution of a weak problem (2.3) with right side $f^*(\cdot) := a(u,\cdot)$. The Ritz projection $P_h : V \to V_h$ is defined via the associated unique discrete solution u_h of (2.4): $P_h u := u_h$, i. e. $P_h u \in V_h$ is the unique solution of

$$a(P_h u, v) = a(u,v), \text{ i. e., } a(u - P_h u, v) = 0 \quad \text{for all } v \in V_h. \tag{2.5}$$

Linear operator P_h is bounded because of (2.1), (2.5), and (2.2)

$$c\|P_h u\|_V^2 \le a(P_h u, P_h u) = a(u, P_h u) \le C\|u\|_V \|P_h u\|_V,$$

i. e. $\|P_h u\|_V \le \frac{C}{c}\|u\|_V$. Also, P_h is a projection: $P_h v = v$ for all $v \in V_h$.

For a Banach space X let ($s \in \mathbb{N}_0 := \{0,1,2,3,\dots\}$, $[a,b] \subset \mathbb{R}$, the set of real numbers)

$$C^s([a,b],X) := \{u : [a,b] \to X : u \text{ is } s\text{-times continuously differentiable}\},$$

$C([a,b],X) := C^0([a,b],X)$, be a space of abstract functions with norm

$$\|u\|_{C^s([a,b],X)} := \sum_{j=0}^{s} \left[\sup_{t\in[a,b]} \|u^{(j)}(t)\|_X \right].$$

We call the X-valued (i. e. abstract) function u differentiable with derivative u' in a point t_0 iff $\lim_{t\to t_0} \|[u(t) - u(t_0)]/[t - t_0] - u'(t_0)\|_X = 0$. For real valued functions $u : [a,b] \to X = \mathbb{R}$ this strong definition becomes the original definition of a derivative.

In the same fashion the Riemann integral of an abstract function can be defined such that the usual rules and the fundamental theorem hold true (see [16, p. 298] or [15, pp. 2–7]). For example, if $u \in C^1([a,b],X)$ then

$$\int_a^b u'(t)\,dt = u(b) - u(a) \in X.$$

It can be easily shown that $C^s([a,b],X)$ is a Banach space because of the completeness of X.

For the direct estimate we need an averaged modulus of continuity for abstract functions (cf. [19, p. 7]). To this end, the r-th difference, $r \in \mathbb{N}$, of a function u at point y is defined as

$$\Delta_\nu^1 u(y) := u(y+\nu) - u(y), \quad \Delta_\nu^r u(y) := \Delta_\nu^1 \Delta_\nu^{r-1} u(y),\ r > 1.$$

For $\delta > 0$ the r-th averaged modulus of continuity of u is given as

$$\tau_r(\delta, u, C([a,b],X)) := \int_a^b \left[\sup\{\|\Delta_\nu^r u(y)\|_X : y, y+r\nu \in [t-\delta, t+\delta] \cap [a,b]\} \right] dt.$$

In this paper we only deal with a first averaged modulus ($r = 1$).

Please note that for $u \in C([a,b],X)$ the integrand is continuous and the real-valued Riemann integral is well defined. Let us remark that a Riemann integrable abstract function u does not need to be continuous almost everywhere (cf. [13]). Therefore, by replacing $|\cdot|$ with $\|\cdot\|_X$ the results of [11] cannot be readily used to show that the integrand of an averaged modulus is Riemann integrable on $[a,b]$ for integrable but non-continuous functions u.

3 Discretization and direct estimate

Instead of (1.1), we deal with the following weak problem ($T > 0$): Find a function $u \in C([0,T],V) \cap C^1([0,T],H)$ that fulfills

$$\begin{aligned} (u_t(t), v)_H + a(u(t), v) &= (f(t), v)_H \quad \text{for all } v \in V,\ t \in [0,T], \\ u(0) &= v_0, \end{aligned} \tag{3.1}$$

where the right side is determined by $f : [0,T] \to H$ and an initial value $v_0 \in V$ is given. Here we denote the first derivative of u by u_t. Beyond it, the notation u_{tt} will be used for the second derivative with respect to the time variable.

For theorems dealing with existence and uniqueness of solutions see [14, Chapter 2], [18, Chapter 11], and [9, Chapter 7.1.1].

Our approach to counterexamples is to find functions $u \in C^1([0,T],V)$ with suitable approximation properties that in a second step can be seen as solutions of problems (3.1). The counterexamples will be chosen such that the right side of (3.1) indeed has a representation $(f(t),\cdot)_H$. But for intermediate results it seems appropriate to replace the right side for each $t \in [0,T]$ by more general bounded linear functionals $f^*_{(t)}$ on V that might not have a representation via the inner product of H. Please note that we do not need to find a solution $u(t)$ of (3.1) for a given functional $f^*_{(t)}$. Vice versa, the functional will be defined by a given "solution" $u(t)$.

Throughout this paper we deal with the following discretization of (3.1). We replace the derivative by a backward difference on the mesh $Z_k := \{0,k,2k,3k,\dots\} \cap [0,T]$ where $k > 0$ denotes the distance of consecutive points: $\overline{\partial_t} u(t) := (u(t) - u(t-k))/k$. Also, we replace V by (finite dimensional) subspaces $\emptyset \neq V_h \in V$, $h \in (0,1]$ (finite element spaces). Additionally, we assume that $V_{h_1} \subset V_{h_2}$ for $h_2 < h_1$. For example, this can be obtained by refining triangulations where h denotes a parameter describing the size of the triangles.

Now we are looking for a function $u_{k,h} : Z_k \to V_h$ such that for all $v \in V_h$, $0 < t \in Z_k$:

$$\begin{aligned} (\overline{\partial_t} u_{k,h}(t), v)_H + a(u_{k,h}(t), v) &= (f(t), v)_H, \\ a(u_{k,h}(0), v) &= a(v_0, v). \end{aligned} \tag{3.2}$$

Using the Ritz projection P_h (see (2.5)) we can write the initial condition as $u_{k,h}(0) = P_h v_0$.

There exists a unique solution $u_{k,h}$ that can be computed iteratively beginning with $u_{k,h}(0) = P_h v_0$ via the recursion

$$(u_{k,h}(t), v)_H + ka(u_{k,h}(t), v) = (u_{k,h}(t-k), v)_H + k(f(t), v)_H$$

for all $v \in V_h$, $0 < t \in Z_k$, because for

$$f^*(v) := (u_{k,h}(t-k) + kf(t), v)_H$$

and the coercive, bounded, and symmetric bilinear form

$$\hat{a}(v,w) := (v,w)_H + ka(v,w)$$

$u_{k,h}(t)$ is the unique solution of the weak problem

$$\hat{a}(u_{k,h}(t), v) = f^*(v) \quad \text{for all } v \in V_h.$$

Please note that the discrete problem (3.2) still has a unique solution if one replaces $(f(t), \cdot)_H$ by a more general bounded linear functional $f^*_{(t)}$ on V_h, as mentioned before.

Theorem 3.1 *Let $u \in C^1([0,T], V)$ be a solution of problem (3.1) where the right side might be a bounded linear functional $f^*_{(t)}$ on V for each $t \in [0,T]$ (instead of $(f(t), \cdot)_H$) and let $u_{k,h}$ be a solution of the corresponding discretization (3.2). Then for each $k > 0$ and $h \in (0,1]$ there holds true:*

$$\begin{aligned} &\max_{t \in Z_k} \|u(t) - u_{k,h}(t)\|_H \\ &\leq \tau_1(k, u_t, C([0,T], H)) + 2 \int_0^T \|u_t(t) - P_h(u_t(t))\|_H \, dt + \|v_0 - P_h v_0\|_H. \end{aligned}$$

If v_0 and $u_t(t)$ belong to V_h then on the right side only the averaged modulus remains. Obviously, this also is the case if $V = V_h$.

The error is measured in terms of the smoothness of solution u. The smoothness is determined by the regularity of v_0 and f, and vice versa (see [9, Chapter 7.1.3]).

For the sake of completeness we prove the direct estimate following [20, p. 12].

Proof.

$$u_{k,h}(t) - u(t) = [u_{k,h}(t) - P_h u(t)] + [P_h u(t) - u(t)] =: e_1(t) + e_2(t).$$

We put $e_1(t)$ into the left side of (3.2) and get utilizing (3.2), (3.1), and (2.5) for $v \in V_h$, $0 < t \in Z_k$:

$$\begin{aligned} &\left(\overline{\partial_t} e_1(t), v\right)_H + a(e_1(t), v) \\ &= \left(\overline{\partial_t} u_{k,h}(t), v\right)_H + a(u_{k,h}(t), v) - \left(\overline{\partial_t} P_h u(t), v\right)_H - a(P_h u(t), v) \\ &= f^*_{(t)}(v) - \left(\overline{\partial_t} P_h u(t), v\right)_H - a(P_h u(t), v) \\ &= \left(u_t(t), v\right)_H + a(u(t), v) - \left(\overline{\partial_t} P_h u(t), v\right)_H - a(P_h u(t), v) \end{aligned}$$

$$
\begin{aligned}
&= \Big(u_t(t), v\Big)_H - \Big(\overline{\partial_t} P_h u(t), v\Big)_H \\
&= \Big(u_t(t) - \overline{\partial_t} u(t), v\Big)_H + \Big(\overline{\partial_t}[u(t) - P_h u(t)], v\Big)_H \\
&= \Big(u_t(t) - \overline{\partial_t} u(t) - \overline{\partial_t} e_2(t), v\Big)_H .
\end{aligned}
$$

We can now choose $v = e_1(t) \in V_h$. That is a reason for decomposing the error into $e_1(t)$ belonging to V_h and a rest that is easy to handle.

$$
\begin{aligned}
&\frac{1}{k}\Big(e_1(t) - e_1(t-k), e_1(t)\Big)_H = \Big(\overline{\partial_t} e_1(t), e_1(t)\Big)_H \\
&\quad = \Big(u_t(t) - \overline{\partial_t} u(t) - \overline{\partial_t} e_2(t), e_1(t)\Big)_H - a(e_1(t), e_1(t)) \\
&\quad \leq \Big(u_t(t) - \overline{\partial_t} u(t) - \overline{\partial_t} e_2(t), e_1(t)\Big)_H ,
\end{aligned}
$$

such that

$$
\frac{1}{k}\|e_1(t)\|_H^2 = \frac{1}{k}\Big(e_1(t-k), e_1(t)\Big)_H + \Big(u_t(t) - \overline{\partial_t} u(t) - \overline{\partial_t} e_2(t), e_1(t)\Big)_H ,
$$

$$
\|e_1(t)\|_H \leq \|e_1(t-k)\|_H + k\|u_t(t) - \overline{\partial_t} u(t)\|_H + k\|\overline{\partial_t} e_2(t)\|_H .
$$

For $j \in \mathbb{N}$, $jk \in Z_k$, we get (note that $\|e_1(0)\|_H = \|u_{k,h}(0) - P_h v_0\|_H = 0$)

$$
\|e_1(jk)\|_H \leq k\sum_{l=1}^{j} \|u_t(lk) - \overline{\partial_t} u(lk)\|_H + k\sum_{l=1}^{j} \|\overline{\partial_t} e_2(lk)\|_H . \tag{3.3}
$$

The first sum can be estimated by an averaged modulus of continuity ($jk \leq T$):

$$
\begin{aligned}
& k\sum_{l=1}^{j} \|u_t(lk) - \overline{\partial_t} u(lk)\|_H \\
&= k\sum_{l=1}^{j} \left\| u_t(lk) - \frac{1}{k}\int_{(l-1)k}^{lk} u_t(t)\,dt \right\|_H \leq \sum_{l=1}^{j} \int_{(l-1)k}^{lk} \|u_t(lk) - u_t(t)\|_H \,dt \\
&\leq \sum_{l=1}^{j} \int_{(l-1)k}^{lk} \Big[\sup\{\|u_t(t_1) - u_t(t_2)\|_H : t_1, t_2 \in [t-k, t+k] \cap [0,T]\}\Big]\,dt \\
&\leq \int_0^T \Big[\sup\{\|u_t(t_1) - u_t(t_2)\|_H : t_1, t_2 \in [t-k, t+k] \cap [0,T]\}\Big]\,dt
\end{aligned}
$$

$$= \quad \tau_1(k, u_t, C([0,T], H)). \tag{3.4}$$

Since P_h is continuous, for $u \in C^1([0,T], V)$ there holds true: $(P_h u(t))_t = P_h(u_t(t))$ and $P_h(u_t(t)) \in C([0,T], V)$. Therefore

$$\overline{\partial}_t e_2(lk) = \frac{1}{k}\int_{(l-1)k}^{lk} [(P_h u(t))_t - u_t(t)]\, dt = \frac{1}{k}\int_{(l-1)k}^{lk} [P_h(u_t(t)) - u_t(t)]\, dt,$$

$$k\sum_{l=1}^{j} \|\overline{\partial}_t e_2(lk)\|_H \le \int_0^T \|u_t(t) - P_h(u_t(t))\|_H\, dt. \tag{3.5}$$

(3.3)–(3.5) establish the error bound for e_1:

$$\|e_1(jk)\|_H \le \tau_1(k, u_t, C([0,T], H)) + \int_0^T \|u_t(t) - P_h(u_t(t))\|_H\, dt.$$

Finally, we estimate e_2:

$$\begin{aligned} \|e_2(t)\|_H &= \|P_h u(t) - u(t)\|_H \\ &= \left\| P_h u(0) - u(0) + \int_0^t (P_h u(y))_t - u_t(y) dy \right\|_H \\ &\le \|v_0 - P_h v_0\|_H + \int_0^T \|u_t(y) - P_h(u_t(y))\|_H dy. \end{aligned}$$ ∎

This direct estimate serves as a simple example for our approach to show sharpness on the basis of counterexamples. Second order approximations like a Crank-Nicolson discretization (cf. [20, p. 14]) would lead to a better rate of convergence but can be analyzed along the same methods (cf. [7] for finite difference schemes).

For real-valued, continuous functions a weak type inequality between the averaged modulus of continuity and an integral modulus of continuity holds true (see [19, p. 18] and the literature cited there, cf. [12]) so that the integral modulus can be used as a measure of smoothness instead of the averaged modulus. This could be investigated for abstract functions as well.

The term $2\int_0^T \|u_t(t) - P_h(u_t(t))\|_H\, dt + \|v_0 - P_h v_0\|_H$ describes the error of Ritz projections, i. e. it shows the influence of spaces V_h. Under natural structural assumptions on the underlying domain (i.e. for open, bounded domains with Lipschitz boundaries) and for typical finite element spaces V_h this error can be estimated against moduli of smoothness by well-known K-functional techniques. Also, the sharpness of such estimates can be established by the means of resonance principles (cf. [10] and the work of Lüttgens cited there).

4 Sharpness

In this section, we show that Theorem 3.1 is sharp in the sense of counterexamples. That is, with Theorem 4.1 we prove that for certain rates of convergence there is a solution of a problem (3.1) such that the error does not vanish faster than the given rate, but at the same time the right side in the estimate of Theorem 3.1 is this rate. Our counterexamples are obtained from a quantitative extension of the uniform boundedness principle (Theorem 4.2). It can handle composed error bounds and is not restricted to the given application in differential equations. However, one needs a context specific lower estimate of the approximation error: Here, the main idea is to express both discrete solutions and the error by the means of discrete Green's functions known from the theory of difference schemes (see Lemma 4.4 and 4.5). To establish an analogue to discrete Green's functions for our setting, we utilize the properties of certain eigenfunctions belonging to the elliptic part of the weak problem (cf. (4.16)). For a particular discrete solution we are able to compute a lower estimate for a seminorm of the discrete Green's function (see Lemma 4.6). In turn, this lower estimate can be applied to the error representation (Lemma 4.5) and it gives the lower bound as required.

In Approximation Theory rates of convergence often are determined by Lipschitz classes. To this end, an abstract modulus of continuity is a function ω, continuous on $[0, \infty)$ such that, for $0 < \delta_1, \delta_2$,

$$0 = \omega(0) < \omega(\delta_1) \leq \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2). \tag{4.1}$$

Functions $\omega(\delta) := \delta^\beta$, $0 < \beta \leq 1$, that are used in Lipschitz classes, satisfy these conditions.

The main result of this paper is the following proposition.

Theorem 4.1 *For each decreasing sequence $(\gamma_n)_{n=1}^\infty \subset (0, 1]$ there exists a strictly decreasing null sequence $(\delta_n)_{n=1}^\infty$ such that the error estimate Theorem 3.1 with parameters $h = \gamma_n$ and $k = \delta_n$ (thus h and k are coupled) is sharp in the following sense:*

For each abstract modulus of continuity ω satisfying

$$\lim_{\delta \to 0+} \frac{\omega(\delta)}{\delta} = \infty \tag{4.2}$$

there exists a counterexample $u_\omega \in C^1([0, T], V)$, that is the solution of a problem (3.1), such that ($\delta \to 0+$, $n \to \infty$, $h = \gamma_n, k = \delta_n \to 0+$)

$$\tau_1(\delta, (u_\omega)_t, C([0, T], H)) = \mathcal{O}(\omega(\delta)), \tag{4.3}$$

$$2\int_0^T \|(u_\omega)_t(t) - P_h((u_\omega)_t(t))\|_H\,dt + \|u_\omega(0) - P_h u_\omega(0)\|_H = \mathcal{O}(\omega(k)), \quad (4.4)$$

$$\max_{t\in Z_k} \|u_\omega(t) - (u_\omega)_{k,h}(t)\|_H \neq o(\omega(k)). \quad (4.5)$$

As mentioned, this result can be obtained as an application of a quantitative version of the uniform boundedness principle. We cite this theorem, add an extension and then demonstrate how to choose parameters in our situation.

For a (real) Banach space X with norm $\|\cdot\|_X$ let $X^\sim$ be the set of non-negative-valued sublinear bounded functionals T on X, i.e., T maps X into $\mathbb{R}$ such that for all $u, v \in X$, $c \in \mathbb{R}$

$$Tu \geq 0, \qquad T(u+v) \leq Tu + Tv, \qquad T(cu) = |c|Tu,$$

$$\|T\|_{X^\sim} := \sup\{Tu : \|u\|_X \leq 1\} < \infty.$$

Theorem 4.2 *(see [3], cf. [5]) Let X be a real Banach space. Suppose that for a sequence of remainders $(T_n)_{n=1}^\infty \subset X^\sim$, and for a measure of smoothness $\{S_\delta : \delta > 0\} \subset X^\sim$ there are test elements $u_n \in X$ such that ($\delta > 0, n \to \infty$)*

$$\|u_n\|_X \leq C_1 \qquad \text{for all } n \in \mathbb{N}, \quad (4.6)$$

$$S_\delta u_n \leq C_2 \min\left\{1, \frac{\sigma(\delta)}{\varphi_n}\right\} \qquad \text{for all } n \in \mathbb{N},\ \delta > 0, \quad (4.7)$$

$$\limsup_{n\to\infty} T_n u_n > 0, \quad (4.8)$$

where $\sigma(\delta)$ is a function, strictly positive on $(0,\infty)$ and $(\varphi_n)_{n=1}^\infty \subset \mathbb{R}$ is a strictly decreasing sequence with $\lim_{n\to\infty} \varphi_n = 0$. Then for each modulus ω satisfying (4.1) and (4.2), there exists a counterexample

$$u_\omega = \sum_{m=1}^\infty \omega(\varphi_{n_m}) u_{n_m} \in X \quad (4.9)$$

with

$$\sum_{m=j}^\infty \omega(\varphi_{n_m}) \leq 2\omega(\varphi_{n_j}), \qquad \sum_{m=1}^\infty \omega(\varphi_{n_m})\|u_{n_m}\|_X < \infty \quad (4.10)$$

(where $(n_m)_{m=1}^\infty \subset \mathbb{N}$ is strictly increasing) such that ($\delta \to 0+$, $n \to \infty$)

$$S_\delta u_\omega = \mathcal{O}\left(\omega(\sigma(\delta))\right), \quad (4.11)$$

$$T_n u_\omega \neq o\left(\omega(\varphi_n)\right). \quad (4.12)$$

(4.12) is equivalent to the existence of a constant $c > 0$ such that

$$\limsup_{n\to\infty} \frac{T_n u_\omega}{\omega(\varphi_n)} > c > 0.$$

In our application, the error bound consists of a measure of smoothness S_δ and a sequence of sublinear bounded functionals $(R_n)_{n=1}^\infty \subset X^\sim$ (cf. (4.4)). It turns out that these functionals satisfy

$$\begin{aligned} \|R_n\|_{X^\sim} &\le C_3 \quad \text{for all } n \in \mathbb{N}, && (4.13)\\ R_n u_j &= 0 \quad \text{for all } 0 < j \le n, \quad n \in \mathbb{N}. && (4.14) \end{aligned}$$

Under these additional assumptions, the counterexample u_ω also fulfills ($n \to \infty$)

$$R_n u_\omega = \mathcal{O}(\omega(\varphi_n)). \tag{4.15}$$

This extension to [3] directly follows from the definition (4.9) of u_ω, (4.14), boundedness conditions (4.6) and (4.13), and (4.10):

$$\begin{aligned} R_n u_\omega &\le \sum_{m=1}^\infty \omega(\varphi_{n_m}) R_n u_{n_m} = \sum_{m:n_m>n}^\infty \omega(\varphi_{n_m}) R_n u_{n_m} \\ &\le \sum_{m:n_m>n}^\infty \omega(\varphi_{n_m}) \|R_n\|_{X^\sim} \|u_{n_m}\|_X \le C_1 C_3 \sum_{m:n_m>n}^\infty \omega(\varphi_{n_m}) \le 2C_1C_3\omega(\varphi_n). \end{aligned}$$

The extension (4.15) is later needed to focus on the error resulting from the difference in time such that the error in space can be ignored. It might serve as a more general concept for error bounds that are composed of several different terms. It allows to proof sharpness restricted to only a subset of such terms.

Theorem 4.2 encapsulates a gliding hump argument where the hump is constructed using the resonance condition (4.8). Therefore, finding suitable resonance elements u_n is crucial. They should be chosen such that it becomes easy to calculate corresponding discrete solutions. To this end, we use eigenfunctions Ψ_h of the bilinear form $a(\cdot,\cdot)$ that has to be symmetric for following considerations. Heat equation (1.1) is an example, where the weak formulation makes use of such a symmetric bilinear form.

The norm $\|u\|_a := \sqrt{a(u,u)}$ is equivalent to $\|\cdot\|_V$ in V and in finite element subspaces $V_h \subset V$. Indeed, $(V, a(\cdot,\cdot))$ and $(V_h, a(\cdot,\cdot))$ are Hilbert spaces. For $f \in V_h$ the functional $f^*(\cdot) := (f,\cdot)_H \in (V_h, (\cdot,\cdot)_V)^* \cong (V_h, a(\cdot,\cdot))^*$ can be represented by Riesz theorem via a linear operator $T_h : V_h \to V_h$ that is uniquely determined by $a(T_h f, v) = (f, v)_H$ for all $f, v \in V_h$. T_h is bounded:

$$\begin{aligned} \|T_h f\|_a^2 &= a(T_h f, T_h f) = (f, T_h f)_H \le \|f\|_H \|T_h f\|_H \\ &\le \|f\|_V \|T_h f\|_V \le C\|f\|_a \|T_h f\|_a. \end{aligned}$$

T_h is self adjunct, because $a(T_h f, v) = (f, v)_H = (v, f)_H = a(T_h v, f) = a(f, T_h v)$, and positive definite, because for $f \neq 0$ there is $a(T_h f, f) = (f, f)_H = \|f\|_H^2 > 0$. Therefore, operator T_h has a real eigenvalue λ_h^{-1} that equals the operator norm $\|T_h\|_{[V_h, \|\cdot\|_a]}$:

$$\lambda_h^{-1} = \|T_h\|_{[V_h, \|\cdot\|_a]} := \sup_{0 \neq v \in V_h} \frac{\|T_h v\|_a}{\|v\|_a}.$$

Let $\Psi_h \in V_h$ be an eigenfunction for eigenvalue λ_h^{-1} that is normed such that $\|\Psi_h\|_H = 1$. Then

$$a(\Psi_h, v) = \lambda_h a(T_h \Psi_h, v) = \lambda_h (\Psi_h, v)_H \quad \text{for all } v \in V_h. \tag{4.16}$$

Essential for the proof of sharpness via resonance condition (4.8) is that the set $\{\lambda_h : h \in (0, 1]\}$ is bounded:

In our setting there is $V_{h_2} \subset V_{h_1}$ for $h_1 < h_2$. Using the Ritz projection we get $T_{h_2} v = P_{h_2} T_{h_1} v$ for all $v \in V_{h_2}$, because $P_{h_2} T_{h_1} : V_{h_1} \to V_{h_2}$ such that for all $f, v \in V_{h_2}$ there is (cf. (2.5))

$$a(P_{h_2} T_{h_1} f, v) = a(T_{h_1} f, v) = (f, v)_H = a(T_{h_2} f, v).$$

We have seen that the Ritz projection is a bounded operator. If one equips V and V_h with the norm $\|\cdot\|_a$ that is equivalent to $\|\cdot\|_V$, then the operator norm $\|P_h\|_{[(V, \|\cdot\|_a), (V_h, \|\cdot\|_a)]} := \sup_{0 \neq v \in V} \frac{\|P_h v\|_a}{\|v\|_a}$ becomes one. This immediately follows from Cauchy-Schwarz inequality

$$\|P_h v\|_a^2 = a(P_h v, P_h v) = a(v, P_h v) \le \|v\|_a \|P_h v\|_a,$$

such that $\|P_h v\|_a \le \|v\|_a$, whereas both sides are equal for $v \in V_h$. Therefore, we get

$$\begin{aligned} \lambda_{h_2}^{-1} &= \|T_{h_2}\|_{[V_{h_2}, \|\cdot\|_a]} \le \|P_{h_2}\|_{[(V, \|\cdot\|_a), (V_{h_2}, \|\cdot\|_a)]} \|T_{h_1}\|_{[(V_{h_2}, \|\cdot\|_a), (V_{h_1}, \|\cdot\|_a)]} \\ &= \|T_{h_1}\|_{[(V_{h_2}, \|\cdot\|_a), (V_{h_1}, \|\cdot\|_a)]} \le \|T_{h_1}\|_{[V_{h_1}, \|\cdot\|_a]} = \lambda_{h_1}^{-1}, \end{aligned}$$

so that $\lambda_{h_1} \le \lambda_{h_2} \le \lambda_1$ and

$$0 < \max\{\lambda_h : h \in (0,1]\} = \lambda_1. \tag{4.17}$$

This also implies, that not only $\|\Psi_h\|_H = 1$ but $\|\Psi_h\|_V$ is bounded independently of h, because in view of (2.1), (4.16):

$$\|\Psi_h\|_V^2 \le \frac{1}{c} a(\Psi_h, \Psi_h) = \frac{1}{c}\lambda_h(\Psi_h, \Psi_h)_H = \frac{\lambda_h}{c}\|\Psi_h\|_H^2 = \frac{\lambda_h}{c} \le \frac{\lambda_1}{c}. \tag{4.18}$$

Lemma 4.3 *Let $\Psi_h \in V_h$ be the eigenfunction belonging to the eigenvalue λ_h in (4.16). We define a function $u(t) := g(t)\Psi_h$ where $g \in C^1[0,T]$, the space of real-valued, continuously differentiable functions on $[0,T]$. Then u is solution of a problem (3.1) with $v_0 := u(0)$ and $f(t) = [g'(t) + \lambda_h g(t)]\Psi_h$, i. e. for all $v \in V_h$ and $t \in [0,T]$ there is*

$$(u_t(t), v)_H + a(u(t), v) = [g'(t) + \lambda_h g(t)](\Psi_h, v)_H. \tag{4.19}$$

For each function $w_{k,h}(t) = \tilde{g}_k(t)\Psi_h : Z_k \to V_h$, $\tilde{g}_k : Z_k \to \mathbb{R}$, there holds true for all $v \in V_h$ ($j > 0$, $jk \in Z_k$, $v \in V_h$):

$$(\overline{\partial_t} w_{k,h}(jk), v)_H + a(w_{k,h}(jk), v) = \hat{\partial}_t \tilde{g}_k(jk)(\Psi_h, v)_H, \tag{4.20}$$

where we use the abbreviation

$$\hat{\partial}_t g(t) = \overline{\partial_t} g(t) + \lambda_h g(t) = \frac{g(t) - g(t-k)}{k} + \lambda_h g(t). \tag{4.21}$$

Let $u_{k,h}$ be the discrete solution of (3.2) associated with u. Then $u_{k,h}$ is the product of a function g_k and Ψ_h with ($jk \in Z_k$)

$$g_k(jk) = \frac{1}{1 + \lambda_h k}\Big(k[g'(jk) + \lambda_h g(jk)] + g_k((j-1)k)\Big), \quad g_k(0) = g(0). \tag{4.22}$$

Proof. Both (4.19) and (4.20) directly follow from (4.16). We have to show that $u_{k,h}(t) := g_k(t)\Psi_h$ indeed is a solution of (3.2). In view of (4.22) function g_k is uniquely determined and ($j \in \mathbb{N}$)

$$\begin{aligned} &(1 + \lambda_h k) g_k(jk) - g_k((j-1)k) = k[g'(jk) + \lambda_h g(jk)], \\ &\hat{\partial}_t g_k(jk) := \overline{\partial_t} g_k(jk) + \lambda_h g_k(jk) = g'(jk) + \lambda_h g(jk). \end{aligned} \tag{4.23}$$

By successively applying (3.1), (4.19), (4.23), and (4.20) we get for all $v \in V,\ j \in \mathbb{N},\ jk \in Z_k$:

$$\begin{aligned}(f(jk), v)_H &= (u_t(jk), v)_H + a(u(jk), v) = [g'(jk) + \lambda_h g(jk)](\Psi_h, v)_H \\ &= \hat{\partial}_t g_k(jk)(\Psi_h, v)_H = (\overline{\partial_t} u_{k,h}(jk), v)_H + a(u_{k,h}(jk), v).\end{aligned}$$

Additionally, $u_{k,h}(0) = g_k(0)\Psi_h = g(0)\Psi_h = u(0) = v_0$ so that $u_{k,h}$ is the unique solution of (3.2). ■

For our purposes, estimates become easier if we work with resonance elements of product type $u(t) = g(t)\Psi,\ g : [0,T] \to \mathbb{R}$, that have discrete counterparts $u_{k,h}(t) = g_{k,h}(t)\Psi$ with the same element $0 \neq \Psi \in V_h$. If Ψ does not fulfill eigenfunction condition (4.16) (i. e. there are $v_1, v_2 \in V_h$, $l_1, l_2 \in \mathbb{R}$, $0 \neq l_1 \neq l_2 \neq 0$, with $a(\Psi, v_1) = l_1(\Psi, v_1)_H,\ a(\Psi, v_2) = l_2(\Psi, v_2)_H$), then a corresponding discrete solution might not be of same product type: If we assume that $u(t) = t^2\Psi$ has an associated discrete solution $u_{k,h}(t) = g_{k,h}(t)\Psi$ ($g_{k,h}(0) = 0$), then for $k \neq -\frac{1}{l_{1,2}}$, $v = v_{1,2}$, and $t = k$ (cf. (3.2))

$$\frac{1}{k}[g_{k,h}(k) - g_{k,h}(0)](\Psi, v)_H + g_{k,h}(k)a(\Psi, v) = 2k(\Psi, v)_H + k^2 a(\Psi, v),$$

$$g_{k,h}(k) + k g_{k,h}(k) l_{1,2} = 2k^2 + k^3 l_{1,2}, \quad g_{k,h}(k) = k^2\left[1 + \frac{1}{1 + k l_{1,2}}\right].$$

Since $l_1 \neq l_2$, $g_{k,h}(k)$ is not well-defined. $u_{k,h}(t)$ cannot have product structure $g_{k,h}(t)\Psi$. An eigenfunction Ψ_{h_2} belonging to V_{h_2} in the sense of (4.16) might not be an eigenfunction for V_{h_1}, $h_1 < h_2$. Even if one uses an eigenfunction to get product structure for $u_{k,h_2}(t)$, then $u_{k,h_1}(t)$ might be not of this structure. This is the reason why we will use separate eigenfunctions for each space V_h that will be condensed to one counterexample by the uniform boundedness principle.

Next, we define an analogue to the discrete Green's function for discrete difference schemes that allows an elegant representation of the approximation error.

For Ψ_h in (4.16) let $G_{k,h}(t_1, t_2) : Z_k \times (Z_k \setminus \{0\}) \to V_h$ for a fixed t_2 be defined as the unique solution of the problem (cf. solution of (3.2))

$$\begin{aligned}([\overline{\partial_t} G_{k,h}(\cdot, t_2)](t), v)_H + a(G_{k,h}(t, t_2), v) &= \begin{cases}(\Psi_h, v)_H, & t = t_2 \\ 0, & t \neq t_2\end{cases}, \\ G_{k,h}(0, t_2) &= 0\end{aligned} \tag{4.24}$$

for all $v \in V_h$, $0 < t \in Z_k$. Especially, there is

$$G_{k,h}(t, jk) = 0 \quad \text{for all } jk > t,\ jk \in Z_k \setminus \{0\},\ t \in Z_k.$$

Lemma 4.4 *Let $w_{k,h}(t) := g_k(t)\Psi_h$, $w_{k,h} : Z_k \to V_h$, $w_{k,h}(0) = 0$ (i. e. $g_k(0) = 0$) with Ψ_h and λ_h as defined in (4.16). Then $w_{k,h}$ has the representation*

$$w_{k,h}(t) = \sum_{j\in\mathbb{N}, jk\le t} G_{k,h}(t, jk)[\hat{\partial}_t g_k(jk)] = \sum_{j\in\mathbb{N}, jk\le t} G_{k,h}(t, jk)[\overline{\partial}_t g_k(jk) + \lambda_h g_k(jk)]. \tag{4.25}$$

Proof. We denote the right side of (4.25) by $r(t)$. If we put $w_{k,h}(t)$ as well as $r(t)$ into the left side of (3.2), then we get equality (cf. (4.24), (4.20)):

$$\begin{aligned}(\overline{\partial}_t r(t), v)_H + a(r(t), v) &= \hat{\partial}_t g_k(t)(\Psi_h, v)_H \\ &= (\overline{\partial}_t w_{k,h}(t), v)_H + a(w_{k,h}(t), v)\end{aligned}$$

for all $v \in V_h$, $0 < t \in Z_k$. Additionally, $r(0) = 0 = w_{k,h}(0)$ so that this proof (like the one before) is completed because of the unique solvability of (3.2). ∎

We use this discrete Green's function in the same fashion to write the error as it is done in [7, 8] for finite difference schemes.

Lemma 4.5 *Let $\Psi_h, u, u_{k,h}, g$ and g_k be defined as in Lemma 4.3. Then the error $u - u_{k,h}$ has the representation ($t \in Z_k$)*

$$u(t) - u_{k,h}(t) = \sum_{j\in\mathbb{N}, jk\le t} G_{k,h}(t, jk)[\overline{\partial}_t g(jk) - g'(jk)]. \tag{4.26}$$

Proof. Because of $u(0) - u_{k,h}(0) = 0$ Lemma 4.4 can be used. (4.25) in connection with (4.23) and (4.21) yields for $t \in Z_k$:

$$\begin{aligned}u(t) - u_{k,h}(t) &= \sum_{j\in\mathbb{N}, jk\le t} G_{k,h}(t, jk)[\hat{\partial}_t g(jk) - \hat{\partial}_t g_k(jk)] \\ &= \sum_{j\in\mathbb{N}, jk\le t} G_{k,h}(t, jk)[\hat{\partial}_t g(jk) - g'(jk) - \lambda_h g(jk)] \\ &= \sum_{j\in\mathbb{N}, jk\le t} G_{k,h}(t, jk)[\overline{\partial}_t g(jk) - g'(jk)].\end{aligned}$$

∎

We close our preparations with the next lemma that will establish the lower estimate in the resonance condition (4.8).

Lemma 4.6 *For a given sequence $(\gamma_n)_{n=1}^{\infty} \subset (0,1]$ of suitable values for parameter h, there exists a positive strictly decreasing null sequence $(\delta_n)_{n=1}^{\infty}$ (as stated in Theorem 4.1) such that for each $h = \gamma_n$ and all $0 < k \leq \delta_n$, there holds true*

$$\max_{t \in Z_k} \left\| \sum_{j \in \mathbb{N}, jk \leq t} G_{k,h}(t, jk) \right\|_H \;\; \geq \;\; c > 0, \qquad (4.27)$$

where c is independent of n.

Proof. In Lemma 4.3 choose $g(t) := \frac{1}{\lambda_h}[1 - \exp(-\lambda_h t)]$. Then $g(0) = 0$ and therefore $u(0) = u_{k,h}(0) = 0$. Further g is a solution of the differential equation $g'(t) + \lambda_h g(t) = 1$. Because of (4.23) and (4.25) the discrete solution $u_{k,h}$ belonging to u has the representation

$$u_{k,h}(t) = \sum_{j \in \mathbb{N}, jk \leq t} G_{k,h}(t, jk).$$

On a fixed finite element space V_h, $h = \gamma_n$, direct estimate Theorem 3.1 for $V = V_h$ and $u \in V_h$ gives

$$\max_{t \in Z_k} \|u(t) - u_{k,h}(t)\|_H \leq \tau_1(k, u_t, C([0,T], H)) \to 0 \quad (k \to 0+).$$

This implies that for each $n \in \mathbb{N}$ (and $h = \gamma_n$) there is a $0 < k_n < T/2$ such that for all $0 < k \leq k_n$:

$$\max_{t \in Z_k} \|u_{k,h}(t)\|_H \geq \frac{1}{2} \max_{t \in Z_k} \|u(t)\|_H = \frac{\|\Psi_h\|_H}{2} \max_{t \in Z_k} \left| \frac{1}{\lambda_h}[1 - \exp(-\lambda_h t)] \right|.$$

Function $1 - \exp(-\lambda_h x)$ is increasing and there is an element $t \in Z_k$ with $t \geq T/2$. Together with $\|\Psi_h\|_H = 1$, we conclude

$$\max_{t \in Z_k} \|u_{k,h}(t)\|_H \geq \frac{1}{2} \frac{1}{\lambda_h}[1 - \exp(-\lambda_h T/2)] \geq \frac{1}{2\lambda_1}[1 - \exp(-\lambda_1 T/2)] = c > 0$$

because $0 < \lambda_h \leq \lambda_1$ (cf. (4.17)) and $\frac{1}{2x}[1 - \exp(-xT/2)] > 0$ is decreasing on $(0, \lambda_1]$.

Now $(\delta_n)_{n=1}^{\infty}$ can be chosen as an arbitrary strictly decreasing null sequence that fulfills $0 < \delta_n \leq k_n$. ∎

We are prepared to prove the main result of this paper:

Proof of Theorem 4.1. We use the resonance principle 4.2 with following parameters

$$
\begin{aligned}
X &= C^1([0,T],V), \quad \|\cdot\|_X = \|\cdot\|_{C^1([0,T],V)}, \\
\varphi_n &= \delta_n = k, \quad \sigma(\delta) = \delta, \\
S_\delta v &= \tau_1(\delta, v_t, C([0,T],H)), \\
T_n v &= \max_{t\in Z_k} \|v(t) - v_{k,h}(t)\|_H, \\
u_n(t) &= \frac{k}{2\pi}\sin\left(2\pi\frac{t}{k}\right)\Psi_h \in C^1([0,T],V_h) \subset C^1([0,T],V).
\end{aligned}
$$

Note that functionals T_n are well-defined for each element of $v \in C^1([0,T],V)$ because each v can be seen as a solution of a problem (3.1) with a right side $f^*_{(t)}$ depending on v such that there is a corresponding discrete solution $v_{k,h}$.

Functionals are sublinear and bounded: $T_n, S_\delta \in (C^1([0,T],V))^\sim$ because of Theorem 3.1 (that holds true for right sides $f^*_{(t)}$)

$$
\begin{aligned}
&T_n v \\
&\le \tau_1(k, v_t, C([0,T],H)) + 2\int_0^T \|v_t(t) - P_h(v_t(t))\|_H\,dt + \|v(0) - P_h v(0)\|_H \\
&\le S_k v + [1 + \|P_h\|_{[V,V_h]}]\big[2T\|v_t\|_{C([0,T],V)} + \|v(0)\|_V\big] \le S_k v + C\|v\|_{C^1([0,T],V)}
\end{aligned}
$$

and $S_\delta v \le 2T\|v\|_{C^1([0,T],H)} \le 2T\|v\|_{C^1([0,T],V)}$.

Also, condition (4.6) is fulfilled (cf. (4.18), $n \to \infty$, i. e. $k \to 0+$):

$$
\|u_n\|_{C^1([0,T],V)} \le \left[\frac{k}{2\pi} + 1\right]\|\Psi_h\|_V \le \left[\frac{k}{2\pi} + 1\right]\frac{\lambda_1}{c} = \mathcal{O}(1).
$$

This directly implies $S_\delta u_n \le 2T\|u_n\|_{C^1([0,T],V)} \le C$. Beyond it, the remaining part of Jackson-Bernstein type condition (4.7) is fulfilled also:

$$
\begin{aligned}
S_\delta u_n = &\int_0^T \left[\sup\left\{\left\|\int_{t_2}^{t_1} (u_n)_{tt}(y)dy\right\|_H : t_1, t_2 \in [t-\delta, t+\delta] \cap [0,T]\right\}\right] dt \\
= \ & \|\Psi_h\|_H \cdot \\
&\cdot \int_0^T \left[\sup\left\{\left|\int_{t_2}^{t_1} -\frac{2\pi}{k}\sin\left(2\pi\frac{y}{k}\right)dy\right| : t_1, t_2 \in [t-\delta, t+\delta] \cap [0,T]\right\}\right] dt \\
\le \ & T 2\delta\frac{2\pi}{k} = C\frac{\delta}{k} = C\frac{\sigma(\delta)}{\varphi_n}.
\end{aligned}
$$

It remains to check resonance condition (4.8). With (4.26) and (4.27) we get

$$\begin{aligned}
T_n u_n &= \max_{t\in Z_k}\left\| \frac{k}{2\pi}\sin\left(2\pi\frac{t}{k}\right)\Psi_h - \left(\frac{k}{2\pi}\sin\left(2\pi\frac{t}{k}\right)\Psi_h\right)_{k,h}(t)\right\|_H \\
&= \max_{t\in Z_k}\left\| \sum_{j\in\mathbb{N}, jk\le t} G_{k,h}(t,jk)\left[\overline{\partial_t}\left(\frac{k}{2\pi}\sin\left(2\pi\frac{jk}{k}\right)\right) - \cos\left(2\pi\frac{jk}{k}\right)\right]\right\|_H \\
&= \max_{t\in Z_k}\left\| \sum_{j\in\mathbb{N}, jk\le t} G_{k,h}(t,jk)\right\|_H \ge c > 0
\end{aligned}$$

where c is independent of n. Thus we have shown $\limsup_{n\to\infty} T_n u_n \ge c > 0$.

To sum up, Theorem 4.2 gives a counterexample $u_\omega \in C^1([0,T],V)$ that fulfills (4.3) and (4.5). It is a solution of a problem (3.1) where for each t the right side is a bounded linear functional $f^*_{(t)}(\cdot) = ((u_\omega)_t(t),\cdot)_H + a(u_\omega(t),\cdot)$ on the space V. But the theorem of Hahn-Banach cannot be used to extend the functional to H since $\|\cdot\|_H \le \|\cdot\|_V$ and not vice versa. Therefore, it is not immediately clear, that $f^*_{(t)}(v) = (f(t),v)_H$, $v\in V$, for a suitable function $f:[0,T]\to H$. Nevertheless, this representation holds true because the resonance elements are built on eigenfunctions. To prove it we use the structure of u_ω as a sum (4.9) that converges in $C^1([0,T],V)$ (and in $C^1([0,T],H)$) such that $\left(\sum_{m=1}^\infty \omega(\varphi_{n_m})u_{n_m}\right)_t = \sum_{m=1}^\infty \omega(\varphi_{n_m})(u_{n_m})_t$. With the abbreviation $g_m := \frac{\omega(\varphi_{n_m})}{2\pi}\sin\left(2\pi\frac{t}{\omega(\varphi_{n_m})}\right)$ we get (using continuity of inner products and (4.19))

$$\begin{aligned}
&((u_\omega)_t(t),v)_H + a(u_\omega(t),v) \\
&\quad= \left(\sum_{m=1}^\infty \omega(\varphi_{n_m})(u_{n_m})_t(t), v\right)_H + a\left(\sum_{m=1}^\infty \omega(\varphi_{n_m})u_{n_m}(t), v\right) \\
&\quad= \sum_{m=1}^\infty \omega(\varphi_{n_m})\left[((u_{n_m})_t(t),v)_H + a\left(u_{n_m}(t),v\right)\right] \\
&\quad= \sum_{m=1}^\infty \omega(\varphi_{n_m})[g_m'(t) + \lambda_{\gamma_{n_m}} g_m(t)](\Psi_{\gamma_{n_m}}, v)_H \\
&\quad= \sum_{m=1}^\infty \omega(\varphi_{n_m})((u_{n_m})_t(t) + \lambda_{\gamma_{n_m}} u_{n_m}(t), v)_H
\end{aligned}$$

$$= \left(\sum_{m=1}^{\infty} \omega(\varphi_{n_m}) \left[(u_{n_m})_t(t) + \lambda_{\gamma_{n_m}} u_{n_m}(t) \right], v \right)_H =: (f(t), v)_H.$$

For the last step, where the infinite sum is moved into the continuous inner product, one needs the convergence of the sum in H (with limit $f(t)$). The sum converges because

$$\left\| \omega(\varphi_{n_m}) \left[(u_{n_m})_t(t) + \lambda_{\gamma_{n_m}} u_{n_m}(t) \right] \right\|_H \leq [1 + \lambda_1] \omega(\varphi_{n_m}) \|u_{n_m}\|_{C^1([0,T],V)}$$

and (see (4.10)) $\sum_{m=1}^{\infty} [1 + \lambda_1] \omega(\varphi_{n_m}) \|u_{n_m}\|_{C^1([0,T],V)} < \infty$.

Finally, we need to prove (4.4). Sublinear functionals

$$R_n v := 2 \int_0^T \|v_t(t) - P_h(v_t(t))\|_H \, dt + \|v(0) - P_h v(0)\|_H$$

are bounded independently of n thus fulfilling (4.13):

$$\begin{aligned} R_n v &\leq [1 + \|P_h\|_{[V,V_h]}] \left[2T \|v_t\|_{C([0,T],V)} + \|v\|_{C([0,T],V)} \right] \\ &\leq 2 \max\{2T, 1\} \|v\|_{C^1([0,T],V)}. \end{aligned}$$

Because $(\gamma_n)_{n=1}^{\infty}$ is decreasing, there is $u_j(t), (u_j)_t(t) \in V_{\gamma_j} \subset V_{\gamma_n} = V_h$ for each $t \in [0, T]$ and $1 \leq j \leq n$. Therefore, $u_j(t) - P_h u_j(t) = 0$ and $(u_j)_t(t) - P_h((u_j)_t(t)) = 0$, which gives (4.14) such that all prerequisites of (4.15) hold true. This in turn establishes (4.4) for $k = \delta_n$. ∎

5 Concluding remarks

Please note that it is not required for $(\gamma_n)_{n=1}^{\infty}$ to be a null sequence, although $h = \gamma_n \to 0+$ would be the regular setting in applications. Resonance elements are chosen such that the error regarding the backward difference dominates the error resulting from the structure of spaces V_h. Especially, Theorem 4.1 holds true if the sequence is constant, $\gamma_n = h_0$ for a fixed $h_0 \in \mathbb{R}$. Then all resonance elements have the representation $u_n(t) = g_n(t) \Psi_{h_0}$ with the same eigenfunction $\Psi_{h_0} \in V_{h_0}$. This leads to a counterexample that shares this structure. For this counterexample all terms in the error bound but the averaged modulus are zero. That shows that, for a semi-discretization where $V = V_{h_0}$, the averaged modulus is a sharp error bound as $k \to 0+$. This part of the error bound cannot be improved.

Theorem 4.2 does not allow us to find a counterexample in the case $\omega(\delta) = \delta$ that is excluded by (4.2). In this situation we only give a counterexample for a semi-discretization with $h = h_0$, $V := V_{h_0}$ as discussed before. Thus we do not couple h and k. We do this for simplicity as we have to explicitly compute a lower estimate of the error. In this context $u_\omega(t) := t^2\Psi_{h_0}$ shows the sharpness, where $\Psi_{h_0} \in V_{h_0}$ as in (4.16). We apply Lemma 4.5 ($t \in Z_k$):

$$\begin{aligned} u_\omega(t) - (u_\omega)_{k,h_0}(t) &= \sum_{j\in\mathbb{N}, jk\le t} G_{k,h_0}(t, jk) \left(\frac{(jk)^2 - ((j-1)k)^2}{k} - 2jk \right) \\ &= -k \sum_{j\in\mathbb{N}, jk\le t} G_{k,h_0}(t, jk). \end{aligned}$$

The sharpness follows in connection with Lemma 4.6:

$$\max_{t\in Z_k} \|u_\omega(t) - (u_\omega)_{k,h_0}(t)\|_H \neq o(k),$$

the rate $\mathcal{O}(\delta)$ of the averaged modulus is obvious.

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